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Non-relativistic Coulomb problem in a one-dimensional quantum mechanics

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In memory of the victims of the terrible earthquake in Armenia, December 1988.

Abstract. It is shown that, unlike the model based only on the potential singularity $U(x) = -|x|^{-1}$, the group of hidden symmetry explains not only the double degeneration of the energy spectrum but also the explicit form of the spectrum, wavefunctions and an extra constant of motion analogous to the Runge-Lenz vector.

1. Introduction

According to quantum mechanics, the discrete spectrum of one-dimensional systems is to be non-degenerate (Landau and Lifshitz 1974). Nevertheless there are some exceptions to the rule. This concerns the potentials: (*a*) symmetric with respect to the inversion $x \to -x$; and (*b*) singular at the point $x = 0$, with the nature of this singularity being such that the probability of a particle penetration from the left side into the right one and vice versa equals 0.

For the potentials satisfying these rather common conditions the energetic discrete spectrum is doubly degenate. A particle with a given energy may be either in the right or in the left region, i.e. there are two variants with two different wavefunctions corresponding to them. The above situation can be shown thus (Avakian *et al* 1987): let us consider a one-dimensional system with the interaction potential $U(x) = Kx^2 + \Omega\delta(x)$, i.e. a harmonic oscillator with a delta-shaped addition. The spectroscopy of this system is such that there are levels with interchanging even and odd wavefunctions. The odd levels do not depend on the parameter Ω and even ones do. With the growth of Ω every even level approaches the adjacent odd one, merges with it and produces a doubly degenerate energetic level.

Instead of 2 one may take N identical wells separated by impermeable barriers and obtain n-fold degeneration in one dimension. There is a question whether the model of a singular-symmetrical potential is the only mechanism explaining a degeneration in one dimension.

A quantum system with the potential $U(x) = -e^2|x|^{-1}$, i.e. a one-dimensional hydrogen atom, bears a direct relation to this question (Loudon 1959). In this case a

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model of a singular-symmetric potential takes place, since the integral of the function $|x|^{-1}$ in the range $(-\varepsilon, \varepsilon)$ is divergent and therefore the regions $x<0$ and $x>0$ are isolated from each other (Andrews 1979). Together with this mechanism, a hidden symmetry 0(2) (Davtyan *et al* 1987b, Boya *et al* 1988) produces a double degeneration. Hence, there are two models explaining the same phenomenon. Evidently, these models do not contradict each other. The purpose of this paper is to attract one's attention to the fact that the symmetry O(2), in contrast to the singular-symmetrical potential model, is responsible not only for the double degeneration but also for the whole dynamics of a one-dimensional hydrogen atom, i.e. an explicit spectrum, an explicit mode of wavefunction and an extra constant motion.

The paper is organised as follows. First we substantiate the fact that, in one dimension, the Coulomb potential is really singular in the above sense. We can obtain this by proving the problem of a one-dimensional hydrogen atom, which is identical to the problem of a particle motion in the field $U=(\alpha x+\beta/x)^2$ representing two identical wells, separated by an impermeable barrier. Then we try to ascertain the form of an extra constant of motion, responsible for the double degeneration. We have solved this problem first with the help of a transition to one dimension in the expression of a Runge-Lenz vector, then with the help of an artificial trick resembling the method of separating variables.

From the point of view of a singular-symmetrical model for the double degeneration the concrete type of singularity is not important. It will satisfy us if the barrier between the regions is impermeable. Thus there should exist an extra constant of motion for the fields, different from the Coulomb, but singular. In this connection there arises the question whether it is possible to obtain an explicit form of this invariant quantity by a certain method. We have proved that, for the potentials $u=-|x|^{-1}+G(|x|)$, where $G(|x|)\to$ constant as $x\to 0$ and having no singularities, variables are separated in 'the one-dimensional parabolic coordinates' and the extra constant of motion coincides with the one for the pure Coulomb field. This result speaks in favour of the singular-symmetrical potential model, though it is not out of place here to pay attention to the moment we have spoken about previously. The information obtainable from the knowledge of the extra constant of motion is very poor, as all that can be done here is to explain the double degeneration. There are no hints at such facts as an explicit form of the energy spectrum and the corresponding wavefunctions. These questions are simply solved in the approach based on the hidden 0(2) symmetry. The rest of this paper is devoted to this. To avoid repeating known facts (Davtyan *et al* 1987b) we develop below the approach based on some other ideas (Hylleraas 1932). We have obtained a differential equation of second order describing the behaviour of a one-dimensional hydrogen atom in the momentum space and developed on this basis the scheme of O(2) symmetry, predicting both the fact of the double degeneration and the explicit form of the energy spectrum, wavefunction and extra constant of motion.

2. Coulomb potential as an oscillator analogy in one dimension

We have mentioned above that the potential $U(x)=-e^2|x|^{-1}$ divides the axis x into two isolated semiaxes $x>0$ and $x<0$. This fact itself is non-self-evident (Andrews 1979). Here we give its simple proof using the transformation

$$x=(\operatorname{sgn} u)u^2.$$

This transformation $R^1 \to R^1$ is trivial and degenerate (one-to-one) at the same time in the series of quadratic non-bijective transformations, converting the Coulomb problems in R^2, R^3 and R^5 into oscillator problems in R^2, R^4 and R^8, respectively (Lambert 1988, Davtyan *et al* 1987a).

Together with the substitution

$$\psi(x) = (|u|)^{1/2}\phi(u)$$

this transformation converts the Schrödinger equation ($e = \hbar = \mu = 1$)

$$d^2\psi/dx^2 + 2(E + 1/|x|)\psi = 0$$

for the one-dimensional hydrogen atom into that for a singular oscillator:

$$d^2\phi/du^2 + 2(\varepsilon - V(u))\phi = 0.$$

Here

$$\varepsilon = 4 + (6|E|)^{1/2}$$

$$V(u) = [2(|E|)^{1/2}u + (3/8)^{1/2}1/u]^2.$$

This is well illustrated in figure 1. Thus, in terms of coordinates u we have two wells identical in form separated by impermeable barriers. This has proved the relation of a one-dimensional hydrogen atom to a singular-symmetrical potential. Thus the fact of the double degeneration of the spectrum of the one-dimensional hydrogen atom is demonstrated.

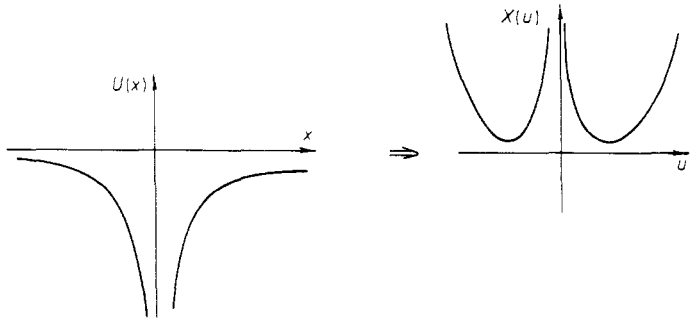

Figure 1.

3. Runge–Lenz vector analogue

It is known from hydrogen atom theory that the Runge–Lenz operator (Englefield 1972) is the constant of motion in the Coulomb field together with the angular momentum

$$\hat{\boldsymbol{A}} = -\frac{\boldsymbol{r}}{r} + \frac{1}{2}\{[\hat{\boldsymbol{p}} \times \hat{\boldsymbol{L}}] - [\hat{\boldsymbol{L}} \times \hat{\boldsymbol{p}}]\}. \tag{3.1}$$

Does there exist a similar constant in one dimension?

In one dimension $\hat{\boldsymbol{L}}=0$ and instead of $\boldsymbol{r}$ in (3.1) one should insert the vector $(x, 0, 0)$. Then only the A_x component will remain different from zero and we denote this by $\hat{A}$. Thus these naive considerations lead us to the quantity

$$\hat{A}=-x/|x|=-\operatorname{sgn} x. \tag{3.2}$$

Before we verify whether the component $\hat{A}$ from (3.2) commutes with the Hamiltonian

$$\hat{\mathscr{H}}=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{1}{|x|} \tag{3.3}$$

we note an important fact. The point $x=0$ does not enter the range of the operator (3.3). This is the so-called regularised Coulomb field which can be obtained from the potential

$$U(x, \alpha)=-\frac{1}{|x|+\alpha}$$

as $\alpha \to 0$. Just in this regularised Coulomb field we have the double degeneration mentioned above. So, we consider further that $x \neq 0$. With this note taken into account we have

$$[\hat{\mathscr{H}}, \hat{A}]=\delta'(x)=0 \tag{3.4}$$

i.e. the operator (3.2) is really a constant of motion. We shall write down the eigenvalue and eigenfunction problem of the operator (3.2)

$$\hat{\mathscr{A}}\psi=A\psi. \tag{3.5}$$

First, from (3.5) it is clear that $A=\pm 1$, as $A^2=1$. Further, the eigenfunction $\psi_{\mathrm{R}}=\theta(x)f(x)$ corresponds to the eigenvalue $A=+1$, and the eigenfunction $\psi_{\mathrm{L}}=\theta(-x)g(x)$ to the eigenvalue $A=-1$ where

$$\theta(x)=\begin{cases}0 & x<0\\ 1 & x>0.\end{cases}$$

As follows from the continuity of the wavefunctions at the point $x=0$, $f(+0)=0$, $g(-0)=0$. The functions ψ_{R} and ψ_{L} are Hamiltonian eigenfunctions (3.3) at $x \neq 0$ and thus $f(x)=g(-x)$. From finiteness of motion it follows that $f(\infty)=g(-\infty)=0$. A complete set of operators $(\hat{\mathscr{H}}, \hat{A})$ corresponds to the basis $(\psi_{\mathrm{R}}, \psi_{\mathrm{L}})$. A qualitative graph of the functions ψ_{R} and ψ_{L} is given in figure 2 (the number of zeros undoubtedly depends on the level number). There is another alternative set $(\hat{\mathscr{H}}, \hat{\mathscr{P}})$, where $\hat{\mathscr{P}}$ is an inversion operator. The basis $(\psi^{(+)}, \psi^{(-)})$:

$$\psi^{(+)}=\frac{1}{\sqrt{2}}(\psi_{\mathrm{R}}+\psi_{\mathrm{L}}) \qquad \psi^{(-)}=\frac{1}{\sqrt{2}}(\psi_{\mathrm{R}}-\psi_{\mathrm{L}}) \tag{3.6}$$

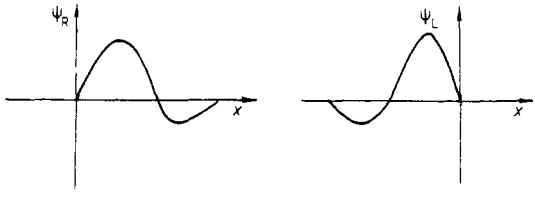

Figure 2.

corresponds to this set. The operators $\hat{A}$ and $\hat{\mathscr{P}}$ anticommute and the commutator $[\hat{\mathscr{P}}_1 \hat{A}] = -2A$ does not lead to a new constant of motion.

4. Speculation on the separation of variables

Can we obtain the operator form (3.2) without turning to three dimensions? We note that in quantum mechanics the Schrödinger equation for a hydrogen atom yields separation of variables in parabolic coordinates. Taking into account this fact we may derive an explicit form of the Z-projection of the operator $\hat{A}$ (Landau and Lifshitz 1974). Projections A_x and A_y can be obtained in the same way if parabolic coordinates are 'oriented' along the axes x and y respectively.

In this section we present some speculative considerations that replace the variable separation method in one dimension. First we note that, to define a particle position in one dimension, it is enough to set the coordinate modulus x and its sign. This method is identical to the definition of the point in two dimensions with the help of polar coordinates. The coordinates $|x|$ and the sign of x are independent. Does there exist a one-dimensional analogue of parabolic coordinates in such a case? In two dimensions parabolic coordinates, 'oriented' along the axis x, look as follows:

$$\mu = \tfrac{1}{2}[(x^2+y^2)^{1/2}+x] \qquad \nu = \tfrac{1}{2}[(x^2+y^2)^{1/2}-x].$$

Hence in the limit $y \to 0$ we have

$$\mu = \theta(x)|x| \qquad \nu = \theta(-x)|x|. \tag{4.1}$$

From (4.1) it follows that

$$\mu + \nu = |x| \qquad \mu - \nu = x. \tag{4.2}$$

In contrast to the coordinates $|x|$ and $\operatorname{sgn} x$ we cannot consider the coordinates (4.1) to be independent. If $\mu \neq 0$ then $\nu = 0$ and vice versa, if $\nu \neq 0$, then $\mu = 0$ (see figure 3). Further, the product $\mu\nu = 0$. At $x \neq 0$ the identity

$$\frac{d^2\psi}{dx^2} = \theta(x)\frac{\partial^2\psi}{\partial\mu^2} + \theta(-x)\frac{\partial^2\psi}{\partial\nu^2} \tag{4.3}$$

is true. From (4.2) it follows that

$$\theta(x) = \frac{\mu}{\mu+\nu} \qquad \theta(-x) = \frac{\nu}{\mu+\nu}.$$

Hence the formulae (4.3) can be represented in terms of the coordinates μ and ν:

$$\frac{d^2\psi}{dx^2} = \frac{\mu}{\mu+\nu}\frac{\partial^2\psi}{\partial\mu^2} + \frac{\nu}{\mu+\nu}\frac{\partial^2\psi}{\partial\nu^2}. \tag{4.4}$$

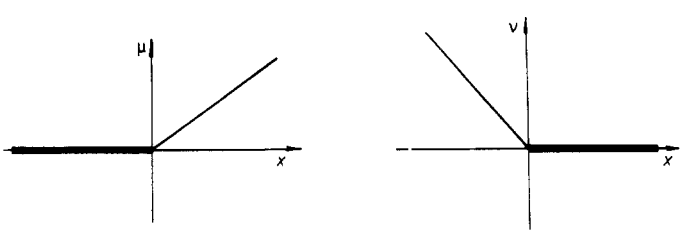

Figure 3.

From (4.4) it follows that the Schrödinger equation is

$$-\frac{1}{2}\frac{\mu}{\mu+\nu}\frac{\partial^2\psi}{\partial\mu^2}-\frac{1}{2}\frac{\nu}{\mu+\nu}\frac{\partial^2\psi}{\partial\nu^2}-\frac{1}{\mu+\nu}\psi=E\psi. \tag{4.5}$$

Let us represent the function ψ as a product $\psi_1(\mu)\psi_2(\nu)$ and proceed in the spirit of the variable separation method. Then, instead of (4.5) we obtain two ordinary differential equations

$$\mu\psi_1''+2E\mu\psi_1+\psi_1=A\psi_1$$

$$\nu\psi_2''+2E\nu\psi_2+\psi_2=-A\psi_2$$

where the parameter A is a separation constant. Eliminating the energy E from these equations we obtain

$$\hat{A}\psi=A\psi$$

where the operator $\hat{A}$ has the form

$$\hat{A}=\frac{\mu\nu}{\mu+\nu}\left(\frac{\partial^2}{\partial\mu^2}-\frac{\partial^2}{\partial\nu^2}\right)-\frac{\mu-\nu}{\mu+\nu}. \tag{4.6}$$

Remembering now that

$$\frac{\mu\nu}{\mu+\nu}=|x|\theta(x)\theta(-x)=0$$

and

$$(\mu-\nu)(\mu+\nu)^{-1}=\operatorname{sgn}x$$

we obtain formula (3.2).

Thus, if we do not pay any attention to the non-correctness of mathematical operations separating formulas (4.5) and (4.6), we may consider that, in one dimension, there exists a method for replacing variable separation in the parabolic coordinates. This method leads to the same result (3.2) as in § 3.

5. Variable separation and a singular symmetrical model

It follows from (3.4) that operator (3.2) commutes $\hat{\mathscr{H}}$ for any potential $U(x)$ and not only in the case of the Coulomb field. How can one understand this? First $[\hat{\mathscr{H}}, \hat{A}]=0$, not at any, but only in a potential singular at the origin of coordinates. Otherwise the regions $x>0$ and $x<0$ are connected and the point $x=0$ should not be neglected. The singular function $\delta'(x)$ appears here and the commutation is violated. Second, $[\hat{\mathscr{H}}, \hat{A}]=0$ for any singular potential, for in this case the point $x=0$ is eliminated from consideration. Thus, the operator (3.2) is a constant of motion for any singular potential. How can this be explained in view of the framework of the method expounded in the previous section?

Let us consider the potential

$$U(x)=-\frac{1}{|x|}+G(|x|). \tag{5.1}$$

Here $G(|x|)$ is a non-singular function at $x=0$, for example $G=|x|$ or $G=x^2$. In terms of the introduced coordinates μ and ν for the Schrödinger equation we have

$$-\frac{\mu}{2}\frac{\partial^2\psi}{\partial\mu^2}-\frac{\nu}{2}\frac{\partial^2\psi}{\partial\nu^2}-\psi-(\mu+\nu)G(\mu+\nu)\psi=(\mu+\nu)E\psi. \tag{5.2}$$

From the equality $\mu\nu=0$ it follows that

$$(\mu+\nu)G(\mu+\nu)=\mu G(\mu)+\nu G(\nu). \tag{5.3}$$

This property can easily be proved by expanding the function $G(\mu+\nu)$ in a series and taking into account that

$$(\mu+\nu)^n=\mu^n+\nu^n.$$

It follows from (5.3) that variables in (5.2) are separated. It is easy to show that for all these, instead of formula (4.6), we obtain the result:

$$\hat{A}=-\frac{\mu\nu}{\mu+\nu}\left(\frac{\partial^2}{\partial\mu^2}-\frac{\partial^2}{\partial\nu^2}\right)+\frac{\mu-\nu}{\mu+\nu}+\frac{4\mu\nu}{\mu+\nu}[G(\mu)+G(\nu)].$$

As $\mu\nu=0$, we again arrive at the constant of motion (3.2). Thus the variable-separation method does not contradict the fact that operator (3.2) is a constant of motion for a singular potential.

6. Hylleraas method

We now consider the problem of a hidden symmetry in a one-dimensional hydrogen atom in more detail. In three dimensions, the hidden symmetry of a hydrogen atom is revealed in the momentum representation (Fock 1935). In this case we have to deal with an integral equation, for in the momentum representation, not a differential but an integral operator corresponds to the modulus of a radius vector $\boldsymbol{r}$. The same is true for the quantity $|x|$ (Davtayn *et al* 1987b). Is there a possibility to describe a hydrogen atom in the momentum representation with the help of a differential equation rather than an integral one? It is clear that for this one should carry out some manipulations in the initial Schrödinger equation, looking to obtain an equation which would contain terms $\boldsymbol{r}$, $\boldsymbol{r}^2$, $\boldsymbol{pr}$ only, i.e. the terms to which a well known ansatz can be applied:

$$\hat{\boldsymbol{p}}=-\mathrm{i}\hbar\nabla\to\boldsymbol{p}\qquad\hat{\boldsymbol{r}}=\mathrm{i}\hbar\nabla_p\to\boldsymbol{r} \tag{6.1}$$

which converts the physics from the coordinate representation into the momentum one. The necessary trick in three dimensions had been invented before a hidden symmetry of the hydrogen atom was discovered (Hylleraas 1932). As applied to one dimension, this trick is as follows.

Let us multiply the Schrödinger equation

$$\left(\frac{\hat{p}^2}{2}-\frac{1}{|x|}\right)\psi=E\psi$$

by $2|x|$ and bring $\chi(x)$ to the form:

$$|x|(p^2-2E)\psi=2\psi. \tag{6.2}$$

We act on equation (6.2) from the left by the operator

$$\hat{Q}=(\hat{p}^2-2E)|x|(\hat{p}^2-2E)$$

and transform $\psi(x)$ to the form

$$X(x) = (\hat{p}^2 - 2E)\psi(x).$$

Then, instead of (6.2), we obtain an equation:

$$(\hat{p}^2 - 2E)|x|(\hat{p}^2 - 2E)|x|X(x) = 4X(x). \tag{6.3}$$

Further, one is easily convinced of the following:

$$|x|\hat{p}^2 - \hat{p}^2|x| = 2\,\mathrm{i}\,\mathrm{sgn}\,x - 2\delta(x).$$

Hence

$$|x|(\hat{p}^2 - 2E)|x| = (\hat{p}^2 - 2E)x^2 - 2\,\mathrm{i}\hat{p}x. \tag{6.4}$$

With (6.4) taken into account we obtain the following equation instead of (6.3):

$$[(\hat{p}^2 - 2E)^2x^2 + 2\mathrm{i}(\hat{p}^2 - 2E)\hat{p}x]X(x) = 4X(x).$$

Now we may use the ansatz (6.1). If we pass from $X(x)$ to its Fourier transform $\phi(p)$, connected with the Fourier transform $a(p)$ of the wavefunction $\psi(x)$ by the relation

$$\phi(p) = (p^2 - 2E)a(p) \tag{6.5}$$

then we arrive at the differential equation

$$(p^2 - 2E)^2\mathrm{d}^2\phi/\mathrm{d}p^2 + 2(p^2 - 2E)p\,\mathrm{d}\phi/\mathrm{d}p + 4\phi = 0. \tag{6.6}$$

We have obtained an equation of the second order. Hence we see that, in comparison with the initial equation (6.2)—of second order—we have neither lost anything nor obtained any new information.

An approach is known from the literature where, instead of (6.6), two differential equations of first order (Nunez Yepez *et al* 1987) are obtained. In spite of its simplicity and elegance, this approach is no consistent as it is based on the ansatz

$$|x| \to \begin{cases} \mathrm{i}\,\mathrm{d}/\mathrm{d}p & x > 0 \\ -\mathrm{i}\,\mathrm{d}/\mathrm{d}p & x < 0. \end{cases} \tag{6.7}$$

It is evident that the rule (6.7) is true only in one case. Consider the functions $R(x)$ and $L(x)$, such that $R(x) = 0$ at $x < 0$ and $L(x) = 0$ at $x > 0$. Then and only then is it easy to prove that

$$\langle R||x||R\rangle = \int_{-\infty}^{\infty} \tilde{R}^*(p)[\mathrm{i}\,\mathrm{d}/\mathrm{d}p]\tilde{R}(p)\,\mathrm{d}p$$

$$\langle L||x||L\rangle = \int_{-\infty}^{\infty} \tilde{L}^*(p)[-\mathrm{i}\,\mathrm{d}/\mathrm{d}p]\tilde{L}(p)\,\mathrm{d}p$$

and thus (6.7) is true. As $\tilde{R}(p)$ and $\tilde{L}(p)$ we take the Fourier transform of the functions $R(x)$ and $L(x)$. This transformation trick (6.7) cannot be applied to any other class of functions. Thus, in this approach it is supposed that the solutions should only be to the class $R(x)$ and $L(x)$. It is not surprising then that the given approach leads to very strange conclusions about the absence of states with a given parity in the one-dimensional hydrogen atom and about the superselection rules (Nunez Yepez *et al* 1988).

7. The hidden symmetry

Equation (6.6), on the face of it, seems to be more complicated than the Schrödinger one (6.2). But it is not so. There is a nice substitution that simplifies equation (6.6) very much. We speak about the substitution (Dartyan *et al* 1987b)

$$p = p_0 \tan(\varphi/2) \qquad -\pi \leqslant \varphi \leqslant \pi \qquad p_0 = (-2E)^{1/2} \tag{7.1}$$

which establishes a one-to-one correspondence between the straight line $(-\infty < p < \infty)$ and the circle with a radius p_0. This substitution is a one-dimensional analogue of a stereographic projection (see figure 4) used in the theory of a hydrogen atom (Bander and Itzykson 1966). Introduce a notation $\phi(p) = G(\varphi)$. It follows from (7.1) that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} = \frac{p^2 + p_0^2}{2p_0} \frac{\mathrm{d}}{\mathrm{d}p}. \tag{7.2}$$

Using (7.2) we write equation (6.6) in terms of the variable φ:

$$\frac{\mathrm{d}^2 G}{\mathrm{d}\varphi^2} + \frac{1}{p_0^2} G = 0. \tag{7.3}$$

We see that the motion of a particle in a one-dimensional Coulomb field is identical to a uniform motion of the particle along a circle with radius p_0. Equation (7.3) has two independent solutions $(n = 1, 2, \ldots)$:

$$G_n^{(R)} = C \exp(-in\varphi) \qquad G_n^{(L)} = C \exp(in\varphi) \tag{7.4}$$

describing a clockwise and counterclockwise rotation, respectively. The energy spectrum is described by the well known Bohr formula:

$$E_n = -1/2n^2, \qquad n = 1, 2, \ldots. \tag{7.5}$$

Another equivalent basis is given by the functions

$$G_n^{(+)} = C \cos n\varphi \qquad G_n^{(-)} = C \sin n\varphi. \tag{7.6}$$

In every state (7.6) the parity, as well as the energy, has a definite value. In fact, the inversion $x \to -x$ is identical to the inversion $p \to -p$ and the last, according to (7.1), is identical to the inversion $\varphi \to -\varphi$.

States (7.4) are eigenfunctions of the generator of the group O(2), i.e. (to the accuracy to a constant) of the operator

$$\hat{L} = -\frac{\mathrm{i}}{n} \frac{\mathrm{d}}{\mathrm{d}\varphi} \tag{7.7}$$

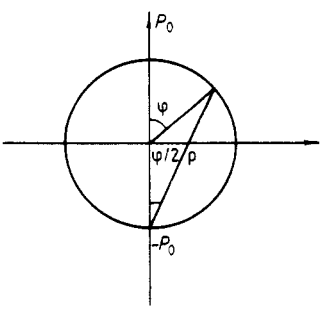

Figure 4.

with the eigenvalues (-1) and $(+1)$ corresponding to the functions $G_n^{(R)}$ and $G_n^{(L)}$, respectively. In the problem of uniform motion along a circle, operator (7.7) plays a fundamental role, as it explains the double degeneration of a spectrum. In other words, it is a constant of motion responsible for the hidden symmetry O(2). It is important that operator (7.7) undoubtedly defines an explicit form of functions (7.4). It is clear that operator (7.7) is connected with the Runge-Lenz operator.

Let us clear up the nature of this connection.

Begin with the equations

$$\hat{L}G_n^{(R)} = -G_n^{(R)} \qquad \hat{L}G_n^{(L)} = G_n^{(L)}. \tag{7.8}$$

We shall transform the variable φ to the variable p. For this purpose we introduce the functions $a_n^{(R)}(p)$ and $a_n^{(L)}(p)$ according to the formulae

$$G_n^{(R)} = (p^2+p_0^2)a_n^{(R)}(p) \qquad G_n^{(L)} = (p^2+p_0^2)a_n^{(L)}(p).$$

Then equations (7.8) can be represented as follows:

$$\hat{A}a_n^{(R)} = -a_n^{(R)} \qquad \hat{A}a_n^{(L)} = a_n^{(L)}$$

where the new operator $\hat{A}$ is defined in the following way:

$$\hat{A} = (p^2+p_0^2)^{-1}\hat{L}(p^2+p_0^2).$$

It follows from (7.1) that

$$\hat{L} = -\frac{\mathrm{i}}{2}\frac{p^2+p_0^2}{p_0 n}\frac{\mathrm{d}}{\mathrm{d}p}.$$

Therefore

$$\hat{A} = -\frac{\mathrm{i}}{2p_0 n}\frac{\mathrm{d}}{\mathrm{d}p}(p^2+p_0^2). \tag{7.9}$$

This operator is an analogue of the Runge-Lenz constant of motion.

Let us convert the simple results given above into less suitable terms of a coordinate representation. From the technical point of view the formula

$$\exp(\mathrm{i}n\varphi) = (-1)^n\left(\frac{p-\mathrm{i}p_0}{p+\mathrm{i}p_0}\right)^n$$

is important here. It follows from this formula that

$$a_n^{(L)}(p) = C(-1)^n\left(\frac{p-\mathrm{i}p_0}{p+\mathrm{i}p_0}\right)^n\frac{1}{p^2+p_0^2}$$

$$a_n^{(R)}(p) = C(-1)^n\left(\frac{p+\mathrm{i}p_0}{p-\mathrm{i}p_0}\right)^n\frac{1}{p^2+p_0^2}.$$

Fourier transforms of these functions are calculated with the method of residues. The result is

$$\psi_n^{(L)} = \text{constant} \times \theta(-x)f(x) \qquad \psi_n^{(R)} = \text{constant} \times \theta(x)f(x) \tag{7.10}$$

where the function $f(x)$ is expressed through the associated Laguerre polynomials

$$f(x) = xL_n^1\left(\frac{2|x|}{n}\right)\exp(-|x|/n).$$

The results of Loudon (1959) can be obtained from this at once, if we choose instead of (7.10) the basis (7.6) with a definite parity.

Let us turn to operator (7.9). According to the analysis (6.1)

$$\hat{A} = \frac{x}{2}\frac{d^2}{dx^2} + xE. \tag{7.11}$$

We shall start with equation (7.2) which can be written down in the following way:

$$\hat{h}\psi = 2\psi$$

where the operator $\hat{h}$ is of the form

$$\hat{h} = -|x|(d^2/dx^2 + 2E).$$

We see that the operator $\hat{A}$ is expressed through the operator $\hat{h}$:

$$A = -\tfrac{1}{2}\operatorname{sgn} x\hat{h}. \tag{7.2}$$

The operators $\hat{A}$ and $\hat{h}$ commute:

$$[\hat{A}, \hat{h}] = -\tfrac{1}{2}(\operatorname{sgn} x\hat{h} - \hat{h}\operatorname{sgn} x)\hat{h} = 0$$

and thus they have common eigenfunctions (7.10). For these functions $h = 2$ and thus (7.2) is transformed into (3.2).

5. Conclusion

Thus, what is more important: the hiden symmetry or the singularity? We have shown that the lauguage of the hiden symmetry possesses an obvious advantage. The hidden symmetry explains not only the fact of the double degeneration, but also the form of spectrum (7.5), the mode of wavefunctions (7.4) and (7.10), a symmetric 'origin' of a constant of motion (7.11) and the condition under which (7.11) transforms into (3.2).

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